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Large N elliptic genus and AdS/CFT correspondence

Jan de Boer

Department of Physics, University of California at Berkeley
366 Le Conte Hall, Berkeley, CA 94720-7300, U.S.A., and
Theoretical Physics Group, Mail Stop 50A–5101
Ernest Orlando Lawrence Berkeley National Laboratory
Berkeley, CA 94720, U.S.A.
E-mail: deboer@thsrv.lbl.gov

Abstract: According to one of Maldacena's dualities, type IIB string theory on $AdS_3 \times S^3 \times K3$ is equivalent to a certain $N = (4, 4)$ superconformal field theory. In this note we compute the elliptic genus of the boundary theory in the supergravity approximation. A finite quantity is obtained once we introduce a particular exclusion principle. In the regime where the supergravity approximation is reliable, we find exact agreement with the elliptic genus of a sigma model with target space $K3^N/S_N$.

Keywords: String Duality, Conformal Field Models in String Theory, Black Holes in String Theory.

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Contents

1. Introduction

One of the prime examples of the AdS↔CFT duality of [1] is the duality between IIB string theory on $M^4 \times S^3 \times AdS_3$ and certain two-dimensional conformal field theories. This duality has its origin in a system of parallel D1 and D5 branes, where the D5 branes are wrapped on some four manifold M^4 which can be either T^4 or $K3$. The two-dimensional conformal field theory in question is the infrared limit of the Higgs branch of the gauge theory on the D1-branes. It has been argued to be a deformation of the $N = (4, 4)$ sigma model with target space $(M^4)^N/S_N$ [2, 3].

There are several interesting issues regarding the precise nature of the sigma model target space and the correspondence between the sigma model moduli and the IIB string theory moduli. These were recently discussed in [4], where (for $M = K3$) it was shown that the sigma model has as target space the singular orbifold $K3^N/S_N$ with vanishing world-sheet theta angle at the singularities. To resolve the metric singularities, one needs to turn on a self-dual NS two-form on $K3$, to change the world-sheet theta angle, one needs to turn on the RR zero and four form field strengths on $K3$. A vanishing world-sheet theta angle was also encountered (for $M = \mathbb{R}^4$) in [5]. Unfortunately, theories with theta angle equal to zero are presumably ill-behaved, and differ considerably from conventional conformal field theory orbifolds, which have theta angle equal to π [6]. Several phenomena are probably closely related to the vanishing of the world-sheet theta angle. One of these is that although according to [5] the Coulomb and Higgs branches of the D1/D5 gauge

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theory decouple, the Higgs branch remains connected to a separate branch coming from a neighborhood of the origin in moduli space [7]. This region corresponds to the throat region of the black hole description of the near extremal NS fivebrane. Another phenomenon is the possibility for sufficiently excited AdS_3 to decay by emitting branes [8]. Both phenomena seem to disappear once we turn on RR zero- and four-form field strengths.

As long as there is no phase transition associated with the vanishing of the world-sheet theta angle, it should be possible to compare BPS quantities obtained from supergravity to those computed at the conformal field theory orbifold point. Previously we showed that Kaluza-Klein spectrum of supergravity reproduces the set of chiral primaries of the $K3^N/S_N$ conformal field theory for large N [9]. In this note we extend this result and show that there is also an exact agreement on the level of the elliptic genus. Some numerical evidence for this was given in [9]. Since the elliptic genus of $K3^N/S_N$ grows linearly with N as $N \to \infty$, we cannot compare the supergravity elliptic genus to that of the $N = \infty$ conformal field theory (both are infinite). In order to make a meaningful comparison, we have to introduce an "exclusion principle" [10] on the supergravity side, so that we can make a comparison for finite and large N. We will determine such an exclusion principle by requiring that the corresponding truncated supergravity spectrum yields the precise set of chiral primaries of $K3^N/S_N$ for finite N. This will be explained in section 2. In sections 3, 4 and 5 we use the exclusion principle to compute the elliptic genus for supergravity, and shows that it agrees with that of the orbifold CFT for states with conformal weight h satisfying $h \leq (N+1)/4$. This includes the regime where supergravity is reliable, but the agreement is valid beyond the supergravity regime until black holes in AdS_3 start to form. Some final comments are given in section 6.

2. An exclusion principle

In [9] the Kaluza-Klein spectrum of six-dimensional $N = (2,0)$ supergravity compactified on $AdS_3 \times S^3$ was computed (see also [11, 12]). This six-dimensional supergravity is obtained by compactifying type IIB string theory on $AdS_3 \times S^3$, and because the size of the $K3$ is much smaller than that of AdS_3 and S^3 the Kaluza-Klein spectrum of six-dimensional supergravity contains the lightest states of type IIB string theory on $AdS_3 \times S^3 \times K3$. The resulting KK spectrum can be organized in representations of the relevant AdS supergroup, which is $\mathrm{SU}(1,1|2)_L \times \mathrm{SU}(1,1|2)_R$. The group $\mathrm{SU}(1,1|2)$ is generated by the global modes $\{L_{\pm 1}, L_0, G^i_{\pm 1/2}, J^i_0\}$ of the $N = 4$ superconformal algebra, and the bosonic part of $\mathrm{SU}(1,1|2)_L \times \mathrm{SU}(1,1|2)_R$ is just the isometry group $\mathrm{SO}(2,2) \times \mathrm{SO}(4)$ of $AdS_3 \times S^3$. The supergroup $\mathrm{SU}(1,1|2)$ has long and short representations, but in the KK spectrum of supergravity only short representations appear. Short representations of $\mathrm{SU}(1,1|2)$ will be denoted by $(j)_S$, and short representations of $\mathrm{SU}(1,1|2)_L \times \mathrm{SU}(1,1|2)_R$ by $(j, j')_S$. A short representation

Multiplicity	States $(k \geq 0)$
1	$\left\|\frac{j}{2}+k, j\right\rangle, \left\|\frac{j}{2}+k, j-2\right\rangle, \ldots, \left\|\frac{j}{2}+k, -j\right\rangle$
2	$\left\|\frac{j+1}{2}+k, j-1\right\rangle, \left\|\frac{j+1}{2}+k, j-3\right\rangle, \ldots, \left\|\frac{j+1}{2}+k, 1-j\right\rangle$
1	$\left\|\frac{j+2}{2}+k, j-2\right\rangle, \left\|\frac{j+2}{2}+k, j-4\right\rangle, \ldots, \left\|\frac{j+2}{2}+k, 2-j\right\rangle$

Table 1: States in the $(j)_S$ short representation.

$(j)_S$ is obtained by taking as highest weight state a chiral primary of the $N=4$ superconformal algebra $|h,j\rangle$ that satisfies $L_0|h,j\rangle = h|h,j\rangle$, $J_0^3|h,j\rangle = j|h,j\rangle$ with $h = j/2$, and by acting on it with $\{L_{\pm 1}, L_0, G^i_{\pm 1/2}, J^i_0\}$. The quantum numbers of the states in such a short representation are shown in table 1, where k is an arbitrary nonnegative integer that appears because of the freedom to act k times with L_{-1} on each state. Notice that each short representation contains precisely one chiral primary. The KK spectrum of six-dimensional supergravity can be organized in the following short representations [9]:

$$\begin{aligned}
&(0,2)_S, (1,3)_S, (2,4)_S, \ldots\,, \\
&21(1,1)_S, 22(2,2)_S, 22(3,3)_S, \ldots\,, \\
&(2,0)_S, (3,1)_S, (4,2)_S, \ldots
\end{aligned} \tag{2.1}$$

The representations $(0,2)_S$ and $(2,0)_S$ contain the higher modes $L_{-(2+k)}$, $G^i_{-(3/2+k)}$, $J^i_{-(1+k)}$, $k \geq 0$, of the $N=4$ superconformal algebra, as one easily sees from table 1. These representations do not correspond to propagating degrees of freedom in the bulk, but to degrees of freedom living only on the boundary (sometimes called "singletons"). For the Virasoro generators this was shown in [13] by relating them to a particular class of diffeomorphisms, and one can easily generalize this to the case with supersymmetry.

The set of (left $\times$ right) chiral primaries of an $N=4$ superconformal field theory can be conveniently encoded in terms of the generalized Poincaré polynomial

$$P_{t,\bar{t}} = \mathrm{Tr}(t^{J_0}\bar{t}^{\bar{J}_0})\,, \tag{2.2}$$

where the trace is taken over the space of chiral primaries only. In case the superconformal field theory is a supersymmetric sigma model with target space M, the Poincaré polynomial equals

$$P_{t,\bar{t}} = \sum_{p,q} h_{p,q} t^p \bar{t}^q\,, \tag{2.3}$$

where $h^{p,q}$ are the Betti numbers of M [14]. The Poincaré polynomial of a resolution of $K3^N/S_N$ called the Hilbert scheme of N points on $K3$ was computed in [15] and has generating function

$$\sum_{N\geq 0} p^N P_{t,\bar{t}}(K3^N/S_N) = \prod_{m=1}^{\infty}(1-p^m t^{m-1}\bar{t}^{m-1})^{-1}(1-p^m t^{m-1}\bar{t}^{m+1})^{-1}\times \\ \times(1-p^m t^{m+1}\bar{t}^{m-1})^{-1}(1-p^m t^{m+1}\bar{t}^{m+1})^{-1}\times \\ \times(1-p^m t^m\bar{t}^m)^{-20}\,. \tag{2.4}$$

We have to compare (2.1) to the set of chiral primaries of the $K3^N/S_N$ SCFT for large N. It is easy to see that $P_{t,\bar{t}}(K3^N/S_N)$ does have a well-defined $N\to\infty$ limit. Because (2.4) has a single factor of $(1-p)^{-1}$, it is of the form

$$a_0+(a_0+a_1)p+(a_0+a_1+a_2)p^2+\cdots=(1-p)^{-1}(a_0+a_1p+a_2p^2+\cdots)\,. \tag{2.5}$$

Thus the $N\to\infty$ limit is obtained by extracting the factor of $(1-p)^{-1}$ and taking $p\to 1$, which yields

$$P_{t,\bar{t}}\left(\frac{K3^\infty}{S_\infty}\right)=(1-t\bar{t})^{-21}\prod_{m=1}^{\infty}(1-t^{m-1}\bar{t}^{m+1})^{-1}(1-t^{m+1}\bar{t}^{m-1})^{-1}(1-t^{m+1}\bar{t}^{m+1})^{-22}\,. \tag{2.6}$$

Since there is a factor $(1-t^j\bar{t}^{j'})^{-1}$ in (2.6) for each short multiplet $(j,j')_S$ in (2.1), the multiparticle chiral primaries of supergravity are in one to one correspondence with those of a sigma model on $K3^\infty/S_\infty$ [9].

It turns out that we can also get an agreement for finite N, once we introduce a suitable exclusion principle. Each of the factors $(1-t^j\bar{t}^{j'})^{-1}$ in (2.6) came from a factor $(1-p^{d(j,j')}t^j\bar{t}^{j'})^{-1}$ in (2.4). If follows then from (2.4) that if we associate degree $d(j,j')$ to the chiral primary in $(j,j')_S$ and keep only those products of chiral primaries which have total degree $\leq N$, we recover precisely the set of chiral primaries of $K3^N/S_N$ for finite N. We will write $(j,j',d)_S$ for a short multiplet to indicate the degree d of the corresponding chiral primary. By comparing (2.6) and (2.4), we find that the table of short multiplets including their degrees reads

$$\begin{array}{rl} (m,m+2;m+1)_S & m=0,1,2,\ldots \\ (m,m;m-1)_S & m=2,3,4,\ldots \\ 20(m,m;m)_S & m=1,2,3,\ldots \\ (m,m;m+1)_S & m=1,2,3,\ldots \\ (m+2,m;m+1)_S & m=0,1,2,\ldots\,. \end{array} \tag{2.7}$$

Although so far we only associated a degree to the chiral primaries, we propose to associate the same degree to all descendants of the chiral primary. At this moment, we don't have a particularly good reason to do so. However, it will turn out that if we extend the exclusion principle to include all descendants of chiral primaries, and keep only products of states whose total degree is $\leq N$, we also find precise agreement for the finite N elliptic genus. Thus, to summarize, for the finite N supergravity Hilbert space we propose the following direct sum of tensor products[1] of $\mathrm{SU}(1,1|2)_L \times \mathrm{SU}(1,1|2)_R$ representations,

$$\mathcal{H}^{(N)}_{\text{sugra}} = \bigoplus_{\substack{\{j_i,j'_i;d_i\} \\ \sum d_i \leq N}} \bigotimes_i (j_i, j'_i; d_i)_S \,. \tag{2.8}$$

It would be interesting to have some independent evidence for the exclusion principle $\sum d_i \leq N$, for instance from conformal field theory.

In the multiparticle Hilbert space (2.8), the generators of the Virasoro algebra appear in a quite asymmetric way. The generators $L_{\pm 1}, L_0$ are part of $\mathrm{SU}(1,1|2)$ and their action does not change the particle number, whereas $L_{-2}, L_{-3}, \ldots$ appear in a separate $\mathrm{SU}(1,1|2)$ representation and their action does change the particle number. In particular, for two chiral primaries A and B, the two states $(L_{-1}A)B$ and $A(L_{-1}B)$ appear as two inequivalent two-particle states in (2.8), whereas $(L_{-2}A)B$ and $A(L_{-2}B)$ appear as two equivalent three-particle states. This is in agreement with conformal field theory, since in conformal field theory $(L_{-1}A)B$ and $A(L_{-1}B)$ are independent, whereas the difference of $(L_{-2}A)B$ and $A(L_{-2}B)$ is proportional to the two-particle state $L^2_{-1}(AB)$. Therefore, $(L_{-2}A)B$ and $A(L_{-2}B)$ should not be counted independently. Another feature of (2.8) is that arbitrary high powers of L_{-1} appear in it, whereas the power of L_{-2} can never be larger than N.

3. Large N elliptic genus of $K3^N/S_N$

The spectrum of left and right-moving chiral primaries is not the only part of the spectrum which is independent of marginal deformations of the theory. A more general object with this property is the elliptic genus, which can only change if a phase transition occurs. The elliptic genus is defined by

$$Z(\tau, z) = \mathrm{Tr}_{RR}(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} y^{J^3_0} \,, \tag{3.1}$$

with $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$, and the trace is over the Ramond sector of the Hilbert space [16, 17, 18]. The elliptic genus receives only contributions from states of the form $|\text{anything}\rangle_L \otimes |\text{groundstate}\rangle_R$, which are 1/4 BPS states of the conformal field theory.

The elliptic genus for $K3$ equals [19, 20]

$$Z(\tau, z) \equiv \sum_{m,l} c(m,l) q^m y^l = 24 \left(\frac{\theta_3(\tau, z)}{\theta_3(\tau, 0)} \right)^2 - 2 \frac{\theta_4(\tau, 0)^4 - \theta_2(\tau, 0)^4}{\eta(\tau)^4} \left(\frac{\theta_1(\tau, z)}{\eta(\tau)} \right)^2 . \tag{3.2}$$

[1] The tensor product of two short representations of $\mathrm{SU}(1,1|2)$ contains one short and many long representations. Therefore, the Hilbert space (2.8) contains many states whose scaling dimensions are not protected by supersymmetry.

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With this definition of $c(m,l)$, the elliptic genus of $K3^N/S_N$ has generating function [21]

$$\sum_{N\geq 0} p^N Z\left(\frac{K3^N}{S_N};\tau,z\right) = \prod_{\substack{n>0\\ m\geq 0,l}} \frac{1}{(1-p^n q^m y^l)^{c(nm,l)}}\,. \tag{3.3}$$

The coefficients $c(m,l)$ are a function of $4m-l^2$ only,

$$c(m,l) \equiv c(4m-l^2)\,, \tag{3.4}$$

and the first few values are $c(r)=0, r<-1$, $c(-1)=2$, $c(0)=20$.

When we try to compare (3.3) to supergravity, we should keep in mind that the supergravity states live in the NS sector of the theory, whereas the elliptic genus (3.1) is defined as a trace over the RR sector. It will therefore be convenient to work with an analogue of the elliptic genus defined directly in the NS sector. Since spectral flow establishes an isomorphism between RR states of the form $|\text{anything}\rangle_L \otimes |\text{groundstate}\rangle_R$ and NS states of the form $|\text{anything}\rangle_L \otimes |\text{chiral primary}\rangle_R$, we can define an NS elliptic genus via

$$Z^{\text{cft}}_{NS}(q,y) = \text{Tr}_{|\text{anything}\rangle_L\otimes|\text{chiral primary}\rangle_R}(-1)^F q^{L_0} y^{J_0^3}\,. \tag{3.5}$$

Spectral flow maps a Ramond state with conformal weight h_R and J_0^3 eigenvalue q_R to an NS state with conformal weight $h_{NS}=c/24+h_R+q_R/2$ and J_0^3 eigenvalue $q_{NS}=q_R+c/6$. It is then straightforward to show that

$$\sum_{N\geq 0} p^N Z^{\text{cft}}_{NS}\left(\frac{K3^N}{S_N};q,y\right) = \prod_{n>0,m,l} \frac{1}{(1-p^n q^m y^l)^{c_{\text{cft}}(n,m,l)}}\,. \tag{3.6}$$

where $c_{\text{cft}}(n,m,l)$ is related to $c(n)$ in (3.4) via

$$c_{\text{cft}}(n,m,l) = c(4mn-n^2-l^2) \tag{3.7}$$

and the product in (3.6) is over m,l that satisfy $2m\in\mathbb{Z}_{\geq 0}$, $m-l/2\in\mathbb{Z}_{\geq 0}$ and $2m\geq |l|$.

In contrast to the Poincaré polynomial considered in the previous section, the NS elliptic genus of $K3^N/S_N$ is not finite in the $N\to\infty$ limit, but diverges linearly as $N\to\infty$. The origin of this linear divergence is the presence of a factor $1/(1-p)^2$ in (3.6). Multiplying $1/(1-p)^2$ with a finite power series in p gives a series whose coefficients diverge linearly for large N,

$$(1-p)^{-2}(a_0+a_1p+a_2p^2+\cdots) = \cdots+((N+1)a_0+Na_1+(N-1)a_2+\cdots)p^N+\cdots. \tag{3.8}$$

Because of this divergence, we cannot simply compare the supergravity result to the conformal field theory result for $N=\infty$. However, thanks to the exclusion principle described above, we can compare the elliptic genus (3.6) for finite N. We should not expect to find a complete agreement though, since supergravity only gives an appropriate description of the spectrum for sufficiently low conformal weights, before string states start to contribute.[2] We will see that there is only agreement if the conformal weight of the states satisfies

$$h \leq \frac{N+1}{4}\,. \tag{3.9}$$

This bound corresponds precisely to the point where the Ramond ground state (which has $h = c/24 = N/4$) appears. A particle with mass m that corresponds to a state with conformal weight $h = \bar{h} \sim N/4$ introduces a deficit angle of 2π in the geometry, and a black hole starts to form (see e.g. [22]). Thus, the supergravity approximation certainly breaks down for $h > N/4$. Interestingly, the exclusion principle yields a proper description of all 1/2 BPS states with conformal weights up to $h \sim N/2$, which overlaps with the black hole regime. On the other hand, it only properly describes the 1/4 BPS states for conformal weights up to $h \sim N/4$. Apparently new 1/4 BPS states but no new 1/2 BPS states appear in the black hole phase.

The part of the elliptic genus that receives only contributions from states with conformal weight satisfying (3.9) has a particularly simple form. Namely, we claim that

$$\begin{aligned}\sum_{N\geq 0} p^N Z^{\rm cft}_{NS}\left(\frac{K3^N}{S_N}; q, y\right) &= \sum_{\substack{n,m,l\\ m\leq (n+1)/4}} p^n q^m y^l (a_{\rm cft}(m,l)n + b_{\rm cft}(m,l)) + \\ &\quad + \sum_{\substack{n,m,l\\ m>(n+1)/4}} (\ldots)\,. \end{aligned}\tag{3.10}$$

The claim follows from (3.8), once we show that for given $(m,l) \neq (0,0)$ there are no $n > 4m$ [3] for which $c_{\rm cft}(n,m,l) = c(4mn - n^2 - l^2) \neq 0$. Since $c(r) = 0$ for $r < -1$, the largest value of n for which $c(4mn - n^2 - l^2) \neq 0$ is always $4m$, thereby proving the claim.

Altogether we have shown that the part of the elliptic genus that can be meaningfully compared to supergravity is the part that behaves as a linear function of N. What we show next is that the supergravity elliptic genus has a decomposition exactly analogous to that in (3.10), with coefficients $a(m,l)$ and $b(m,l)$ identical to those appearing in the conformal field theory elliptic genus.

[2] The first string states that appear are Kaluza-Klein and winding string states coming from the $K3$. Their left and right-moving conformal weights h and $\bar{h}$ satisfy $h + \bar{h} \sim (g_6 Q_5)^{1/2}$ and $h + \bar{h} \sim (g_6 Q_1)^{1/2}$ respectively.

[3] That we have n instead of $n+1$ in this inequality is due to the following fact: if the left hand side of (3.8) truncates at a_r, then the coefficients on the right hand side are linear functions of n for $n \geq r-1$.

4. Large N elliptic genus of supergravity

To define the supergravity elliptic genus we use exactly the same definition as in conformal field theory, taking a trace over the supergravity Hilbert space (2.8) rather than the conformal field theory Hilbert space. Since supergravity describes the NS sector of the theory, we should use the definition of the NS elliptic genus in (3.5). Thus, we define

$$Z_{NS}^{\mathrm{sugra},(N)}(q,y) = \mathrm{Tr}_{|\mathrm{anything}\rangle_L \otimes |\mathrm{chiral\ primary}\rangle_R \in \mathcal{H}^{(N)}_{\mathrm{sugra}}} (-1)^F q^{L_0} y^{J_0^3} \,. \tag{4.1}$$

Due to the multiparticle form of the supergravity Hilbert space, we can write the generating function for the elliptic genera (4.1) in a form similar to (3.6),

$$\sum_{N\geq 0} p^N Z_{NS}^{\mathrm{sugra},(N)}(q,y) = \prod_{n>0,m,l} \frac{1}{(1-p^n q^m y^l)^{c_{\mathrm{sugra}}(n,m,l)}} \,. \tag{4.2}$$

The powers $c_{\mathrm{sugra}}(n,m,l)$ appearing in (4.2) are completely different from the powers $c_{\mathrm{cft}}(n,m,l)$ appearing in (3.6). The number $c_{\mathrm{sugra}}(n,m,l)$ is the number of single particle states of degree n in supergravity, counted with a sign $(-1)^F$, of the form $|\mathrm{anything}\rangle_L \otimes |\mathrm{chiral\ primary}\rangle_R$, where the conformal weight and J_0^3 eigenvalue of $|\mathrm{anything}\rangle_L$ are m and l respectively.[4] It follows from (2.7) that $c_{\mathrm{sugra}}(n,m,l) \leq 48$. On the other hand, the numbers $c_{\mathrm{cft}}(n,m,l)$ grow exponentially. It is therefore quite remarkable that there is any relation at all between (3.6) and (4.2).

To demonstrate this relation, we first show that (4.2) can be decomposed in the same way as in (3.10), namely

$$\begin{aligned}\sum_{N\geq 0} p^N Z_{NS}^{\mathrm{sugra},(N)}(q,y) = &\sum_{\substack{n,m,l\\ m\leq (n+1)/4}} p^n q^m y^l (a_{\mathrm{sugra}}(m,l)n + b_{\mathrm{sugra}}(m,l)) + \\ &+ \sum_{\substack{n,m,l\\ m> (n+1)/4}} (\ldots).\end{aligned} \tag{4.3}$$

The reasoning is similar to that below (3.6). First, we notice that (4.1) has a factor of $1/(1-p)^2$, just like (3.5), which is responsible for the linear growth of the elliptic genus as a function of N. Next, by inspecting the table of KK states (2.7) we see that for any given L_0 eigenvalue m and J_0^3 eigenvalue l, there is at most one KK state with degree $d \geq 4m$. We can now repeat the argument given below (3.10) to establish (4.3).

5. Equivalence of elliptic genera

As we said before, we certainly cannot trust supergravity to give a reliable description of states with conformal weight $h \geq (N+1)/4$. Therefore, if the AdS$\leftrightarrow$CFT duality

[4] Except for $c_{\mathrm{sugra}}(1,0,0)$, which is equal to 2.

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is correct, we should only expect

$$a_{\rm cft}(m,l) = a_{\rm sugra}(m,l)\,, \qquad b_{\rm cft}(m,l) = b_{\rm sugra}(m,l)\,. \tag{5.1}$$

These conditions are much weaker then the equivalence of the full elliptic genera, which would imply $c_{\rm sugra}(n,m,l) = c_{\rm cft}(n,m,l)$ and which is certainly not true. The coefficients $a(m,l)$ and $b(m,l)$ are fixed by the residues of (3.5) and (4.1) at the double pole $p=1$. This can easily be seen from (3.8): If we write the left hand side of (3.8) as $c_{-2}(1-p)^{-2} + c_{-1}(1-p)^{-1} +$ regular, then $a = c_{-2}$ and $b = c_{-2} + c_{-1}$. These residues can be extracted directly from (3.5) and (4.1), by removing the factor of $1/(1-p)^2$ and by evaluating the remainder and its p-derivative at $p=1$. This leads to a useful equivalent description of (5.1), namely we find that (5.1) is valid if and only if the "first two moments" of the $c_{\rm cft}$ and $c_{\rm sugra}$ agree, i.e.

$$\sum_n c_{\rm cft}(n,m,l) = \sum_n c_{\rm sugra}(n,m,l)\,, \qquad \sum_n n c_{\rm cft}(n,m,l) = \sum_n n c_{\rm sugra}(n,m,l)\,. \tag{5.2}$$

This clearly shows that the equivalence of the supergravity part of the elliptic genera is much weaker than the equivalence of the full elliptic genera, which would require the c's themselves to be identical.

In the remainder of this section, we compute the left and right hand sides of (5.2) and show that they are the same.

We first consider $\sum_n c_{\rm cft}(n,m,l)$ and $\sum_n n c_{\rm cft}(n,m,l)$. The two crucial identities that allow us to evaluate these sums are

$$Z_{K3}(q,y)|_{y=1} = 24\,, \qquad \frac{\partial}{\partial y} Z_{K3}(q,y)|_{y=1} = 0\,, \tag{5.3}$$

which follow from (3.2) and $\theta_1(\tau,0)=0$, $\frac{\partial}{\partial z}\theta_3(\tau,z)|_{z=0} = 0$. The identities (5.3) imply that the coefficients $c(m,l) = c(4m-l^2)$ that appear in the expansion of (3.2) obey

$$\sum_l c(4m-l^2) = 24\delta_{m,0} \tag{5.4}$$

$$\sum_l l c(4m-l^2) = 0\,. \tag{5.5}$$

Consider now for example $\sum_n c_{\rm cft}(n,m,l)$. According to (3.7) we get

$$\begin{aligned}\sum_{n>0} c_{\rm cft}(n,m,l) &= \sum_{n>0} c(4nm-n^2-l^2) \\ &= \sum_{n>0} c(4u(t-u)-(n-t)^2)\,, \end{aligned} \tag{5.6}$$

where in the second line we substituted $m=t/2$, $l=t-2u$, with t,u nonnegative integers. The second line can be evaluated using (5.4), where we have to be careful that the sum over n in (5.6) may not reproduce all terms in the sum over l in (5.4)

values for l	$\sum_{n>0} c_{\rm cft}(n,m,l)$	$\sum_{n>0} nc_{\rm cft}(n,m,l)$
$\lvert l\rvert > 1$	$24\delta_{m,\lvert l/2\rvert}$	$48m\delta_{m,\lvert l/2\rvert}$
$\lvert l\rvert = 1$	$24\delta_{m,1/2}-2$	$48m\delta_{m,1/2}$
$l = 0$	$22\delta_{m,0}-20$	$2\delta_{m,0}$

Table 2: Values of $\sum_{n>0} nc_{\rm cft}(n,m,l)$.

(this happens only for $|l| \leq 1$, in which case we also need to use $c(-1) = 2$ and $c(0) = 20$). In the same way, we can evaluate $\sum_{n>0} nc_{\rm cft}(n,m,l)$ using both (5.4) and (5.5). The results are summarized in table 2.

Next we turn to the sums $\sum_{n>0} c_{\rm sugra}(n,m,l)$ and $\sum_{n>0} nc_{\rm sugra}(n,m,l)$. The easiest way to determine these is by first computing the generating function

$$s(p,q,y) = \sum_{n,m,l} c_{\rm sugra}(n,m,l)p^n q^m y^l \tag{5.7}$$

and by evaluating $s(p,q,y)$ and its p-derivative at $p=1$. Since $c_{\rm sugra}(n,m,l)$ counts the number of single particle KK states with particular quantum numbers and degrees, we can construct $s(p,q,y)$ directly from table (2.7), and get

$$\begin{aligned}(1-q)(y-y^{-1})s(p,q,y) = {} & 2p(1-q)(y-y^{-1}) + \\ & + (20p+p^2)(q^{1/2}(y^2-y^{-2}) - 2q(y-y^{-1})) + \\ & + \frac{2p+20p^2+2p^3}{1-pq^{1/2}y}(qy^3 - 2q^{3/2}y^2 + q^2 y) + \\ & + \frac{2p+20p^2+2p^3}{1-pq^{1/2}y^{-1}}(-qy^{-3} + 2q^{3/2}y^{-2} - q^2 y^{-1})\,. \end{aligned} \tag{5.8}$$

The factor of $(1-q)$ in the left hand side has its origin in the fact that single particle states can carry arbitrary powers of L_{-1}, the factor of $y-y^{-1}$ arises because SU(2) representations of spin $j/2$ give a contribution $(y^{j+1}-y^{-j-1})/(y-y^{-1})$.

From (5.8) we obtain

$$\begin{aligned} s(1,q,y) &= \frac{-46+26q-2q^{1/2}(y+y^{-1})}{1-q} + \frac{24}{1-q^{1/2}y} + \frac{24}{1-q^{1/2}y^{-1}} \\ \frac{\partial}{\partial p} s(p,q,y)|_{p=1} &= 2 + \frac{24q^{1/2}y}{(1-q^{1/2}y)^2} + \frac{24q^{1/2}y^{-1}}{(1-q^{1/2}y^{-1})^2}\,. \end{aligned} \tag{5.9}$$

If we expand the results in (5.9) we recover precisely the results of table 2, which proves that the conformal field theory and supergravity elliptic genus are equivalent in the supergravity regime.

6. Comments

In this paper we have shown the equivalence of the conformal field theory and supergravity elliptic genus in the supergravity regime, extending the numerical evidence in [9]. Although the definition of the supergravity elliptic genus involves one piece of information which cannot be obtained purely within supergravity, namely an exclusion principle, we consider the fact that the elliptic genera agree as very strong evidence in favor of both the proposed exclusion principle as well as the AdS↔CFT correspondence. In addition, this completely resolves the puzzle in [23].

Originally, one of the motivations for this work was to examine to what extend supergravity can account for the entropy of black holes. In [3] this entropy was determined by looking at the degeneracy of 1/4 BPS states, which are counted by the elliptic genus. However, the region of interest is the one where the conformal weight is much greater than the central charge, $h \gg 6N$. This is not the region $h \leq (N+1)/4$ for which we proved the equivalence. The supergravity elliptic genus grows much slower for $h \gg 6N$ than the conformal field theory elliptic genus, because $c_{\rm sugra}(n,m,l)$ is bounded, whereas $c_{\rm cft}(n,m,l)$ grows exponentially as a function of m. Thus supergravity, subject to the exclusion principle, cannot account for the entropy of the black hole. On the other hand, if one drops the exclusion principle, the number of supergravity states grows more rapidly than those of the conformal field theory. As a function of energy, the six-dimensional supergravity entropy scales as $E^{5/6}$ but the conformal field theory entropy only as $E^{1/2}$ [24]. Therefore, a more generous exclusion principle might allow one to encode the entropy of the black hole in terms of supergravity states. However, since stringy states are clearly important for the description of the black hole, such a supergravity description would be a convenient parametrization rather than an adequate microscopic description of the states that make up the black hole entropy.

Along these lines, it is intriguing to notice that equations (5.2) have a unique solution for $c_{\rm cft}(n,m,l)$ in terms of $c_{\rm sugra}(n,m,l)$, once we know that $c_{\rm cft}(n,m,l)$ is of the form $c(4mn-n^2-l^2)$. Thus, it is possible to reconstruct the $K3^N/S_N$ elliptic genus from the supergravity data, although it remains to be seen whether this has any physical meaning.

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